One-dimensional magnetogasdynamics in oblique fields

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Earlier work on the dynamics of a perfectly conductive gas in situations where all variables depend on one space co-ordinate only is extended to the case where the magnetic field has a component in the direction of variation. The theory is developed for an arbitrary gas in equilibrium, subject only to certain reasonable restrictions.

The first main section studies the variation of the transverse field component in slow and fast simple waves and the tendency of compressive waves in which the transverse field does not change sign to steepen into shocks.

The next section develops a symmetrical treatment of the Rayleigh, Fanno and other processes of ordinary steady one-dimensional gasdynamics, generalized to allow for electromagnetic effects. The slow, fast and Alfvén wave speeds are critical. A particular case of this analysis is a generalization of ordinary gasdynamics which allows for the effect of transverse forces such as occur in turbomachinery.

The final section is an exhaustive study of shocks in the presence of a field component normal to the shock front. From the generalized Rayleigh line it is established that there are up to six different types of shocks, all compressive and distinguishable by the relative magnitudes of the upstream and downstream normal velocities in comparison with the local slow, fast and Alfvén wave speeds.

Some aspects of shock structure are discussed briefly.

1. Introduction

In an earlier paper (Shercliff 1960) the theory of one-dimensional magnetogasdynamics was developed for arbitrary, isotropic, conducting gases in thermodynamic equilibrium for the case where the magnetic field is transverse to the longitudinal direction, i.e. the direction in which variation occurs. In this paper we extend the analysis to the situation where there is also a longitudinal magnetic field component. This adds the complication that transverse momentum is no longer conserved.

The meanings of some of the less obvious symbols used in the paper are defined in the following list.

Notation

B = magnitude of magnetic field $\mathbf{B}$ $\quad$ B_x, etc., Cartesian components of $\mathbf{B}$

$b = B/(\mu\rho)^{\frac{1}{2}}$ $\quad$ $b_x = B_x/(\mu\rho)^{\frac{1}{2}}$, etc.

c = fast or slow sound speeds (c_f and c_s, respectively)

$F_x = p+\rho v_x^2+B_y^2/2\mu$ $\quad$ $F_y = \rho v_x v_y - B_x B_y/\mu$

$f_x = p+\rho v_x^2$ $\quad$ $f_y = \rho v_x v_y$
$G = \rho v_x$ $\quad$ $H = h+\frac{1}{2}(v_x^2+v_y^2)$
h = enthalpy; also $|\Delta B_y|/B$, upstream of shock
$s_0 = (a/b)^2$, upstream of shock $\quad$ u = internal energy
v = magnitude of velocity $\mathbf{v}$ $\quad$ v_x, etc., Cartesian components of $\mathbf{v}$
$X = v_x^4 - v_x^2(a^2+b^2)+a^2b_x^2$ $\quad$ $\theta_0 = \tan^{-1}(B_y/B_x)$, upstream of shock
$\rho_0 = 1/\tau_0$ $\quad$ $\tau = 1/\rho$, specific volume
$\tau_0 = B_x^2/\mu G^2$ $\quad$ $\tau_x = F_x/G^2$
Δ = difference across shock $\quad$ $\tau_y = F_y/G^2$
$\bar{\tau}$, etc., bar denotes mean between two sides of shock
1, 2 (suffices) upstream, downstream of shock

It is desirable to develop the theory so as to apply to any gas, since magnetogasdynamics often concerns partially ionized or reactive gases of variable mean molecular weight. It is, moreover, easier on the whole to treat a general gas rather than a perfect gas, for which the algebra becomes very complicated. The speed of sound, a, which appears in the analysis, refers to the speed of waves of low enough frequency for the maintenance of equilibrium of ionization and other effects. The only restrictions placed on the properties of the gas are: (i) stability; i.e. $(\partial p/\partial \rho)_s > 0$ and $c_p > 0$, and (ii) the conditions of Weyl (1949); $(\partial^2 p/\partial \tau^2)_s > 0$, and positive expansion coefficient, i.e. $(\partial \tau/\partial T)_p > 0$, $(\partial p/\partial s)_\tau > 0$, etc.

There are three main sections in the paper. Section 2 deals with the behaviour of simple waves in a perfect conductor, investigating particularly the changes of magnitude and sign of the transverse field component and the tendency of compressive waves in which the transverse field does not change sign to steepen into shocks.

Section 3 discusses steady flows in a perfect conductor caused by external influences such as gravity, energy release, etc. These are generalizations of the Rayleigh and other processes of classical one-dimensional gasdynamics. The crucial significance of the two sound speeds, fast and slow, is explored. A degenerate case of these flows is a generalization of the ordinary gasdynamics discussed in a previous communication (Shercliff 1958). Here the ordinary sound speed is dominant.

The final and most important section is an exhaustive but, it is hoped, simple treatment of magnetogasdynamic shocks in the presence of a field component normal to the shock front. For convenience this section is written so as to be intelligible by itself without reference to the rest of the paper. The approach is to exploit the properties of the generalized Rayleigh line, from which it is easy to establish such facts as the possibility of *six* different types of shock, all compressive. The results are reconciled with the work of earlier writers, most of whom performed numerical investigations of the case of a perfect gas with $\gamma = \frac{5}{3}$.

In terms of the distinction between approximate and strict one-dimensionality made in the earlier paper (Shercliff 1960), the motions studied herein are *strictly* one-dimensional. All quantities are strictly independent of the transverse co-ordinates and Maxwell's equations are obeyed. All quantities are assumed to vary sufficiently slowly with distance for finite electrical conductivity to have negligible effect, i.e. the magnetic Reynolds number is assumed to be virtually infinite. This is not true within shocks, of course, but in this paper we are not primarily concerned with shock structure.

We select the x-axis in the longitudinal direction. The equations

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t \quad \text{and} \quad \operatorname{div}\mathbf{B} = 0$$

then indicate that $\partial/\partial t$ and $\partial/\partial x$ of B_x vanish and $B_x = \text{const}$. We shall also assume it positive. In §§ 2 and 3 we restrict the investigation to cases where $\mathbf{B}$ and the fluid velocity $\mathbf{v}$ lie in xy-planes and $B_z = v_z = 0$.

The equation of continuity is

$$\frac{D\rho}{Dt}+\rho\frac{\partial v_x}{\partial x} = 0, \tag{1}$$

and the dynamic equations state that (in MKS units)

$$\rho\frac{Dv_x}{Dt}+\frac{\partial}{\partial x}\left(p+\frac{B_y^2}{2\mu}\right) = 0 \quad \text{and} \quad \rho\frac{Dv_y}{Dt} = \frac{B_x}{\mu}\frac{\partial B_y}{\partial x}, \tag{2}$$

in the absence of viscous stress or any additional force on the fluid. For a perfect conductor we have

$$\frac{DB_y}{Dt} = B_x\frac{\partial v_y}{\partial x}-B_y\frac{\partial v_x}{\partial x}, \tag{3}$$

while in the absence of viscous or Ohmic dissipation, heat exchange or relaxation processes (i.e. thermodynamic disequilibrium) we have

$$\frac{Ds}{Dt} = 0. \tag{4}$$

In steady flow this implies s is constant. We shall assume this is also true in unsteady cases, so that $dp = a^2 d\rho$.

2. Simple waves

It is well known (Kulikovsky 1958; Friedrichs & Kranzer 1958) that simple-wave solutions of the equations (1) to (4) exist, there being two wave speeds, the fast and the slow, in each direction. The Alfvén mode has been excluded by the restriction that v_z and B_z vanish. We remark in passing that Alfvén simple waves are three-dimensional and involve arbitrary rotations of a transverse field component of constant magnitude, the thermodynamic state being constant. Thus the wave speed $B_x/(\mu\rho)^{\frac{1}{2}}$ is constant and there is no steepening or spreading tendency.

For a simple wave in the xy-plane, travelling in the x-direction at a speed c, we may replace D/Dt by $-c\partial/\partial x$ in (1) to (3), which become, in differential form at a given instant in time,

$$\left.\begin{aligned} \rho\, dv_x = c\, d\rho, \quad -\rho c\, dv_x+a^2 d\rho+B_y dB_y/\mu = 0, \\ -\rho c\, dv_y = B_x dB_y/\mu, \quad -c\, dB_y = B_x dv_y-B_y dv_x. \end{aligned}\right\} \tag{5}$$

The eliminant of (5) is the familiar equation which gives the real fast and slow wave speeds

$$c^4-c^2(a^2+b^2)+a^2 b_x^2 = 0, \tag{6}$$

where $b_x = B_x/(\mu\rho)^{\frac{1}{2}}$, $b_y = B_y/(\mu\rho)^{\frac{1}{2}}$ and $b^2 = b_x^2 + b_y^2$. Equation (6) may usefully be rewritten

$$\frac{c^2-b^2}{a^2} = \frac{b_y^2}{c^2-a^2} = \frac{c^2-b_x^2}{c^2}. \tag{7}$$

These three quantities are positive for fast waves, negative for slow waves. From the first two equations (5) it follows that

$$\frac{d\rho}{\rho}\frac{a^2}{c^2-b^2} = \frac{dB_y}{B_y} = \frac{db_y}{b_y} + \frac{1}{2}\frac{d\rho}{\rho}. \tag{8}$$

This shows that compressive fast waves increase $|B_y|$ while compressive slow waves decrease $|B_y|$, the reverse being true for expansive waves. These facts have been pointed out by various authors (e.g. Friedrichs & Kranzer 1958).

A right-travelling simple wave of either type may be specified by an arbitrary distribution of density $\rho(x)$ at some initial instant, the other variables v_x, v_y and B_y then being determined by equations (5). If all four of ρ, v_x, v_y and B_y were chosen arbitrarily at an instant, the resultant motion would in general involve left- and right-travelling slow and fast waves, and these would not be simple waves any longer. Simple waves cannot be superposed as the problem is non-linear.

It is instructive to study the equation which determines B_y, given $\rho(x)$, in the case of simple waves. This can conveniently be obtained by eliminating c from (6) and (8), to give

$$\left(\frac{dB_y}{d\rho}\right)^2 - \left(\frac{B_x^2 + B_y^2 - \mu\rho a^2}{\rho B_y}\right)\frac{dB_y}{d\rho} - \frac{\mu a^2}{\rho} = 0. \tag{9}$$

This gives two real differential equations of the first order. For any values of B_y and ρ, the two values of $dB_y/d\rho$ have opposite signs, their product being $-\mu a^2/\rho$, and the two families of solutions may readily be identified as fast or slow, in accordance with the comment after (8). Typical trajectories of these solutions on the B_y/ρ plane are illustrated in figure 1. The resemblance of the curves to orthogonal confocal conics is striking. Indeed the curves have exactly this form in the case of a perfect gas for which $\gamma = 2$, but the curves have approximately this shape for any gas which satisfies the two conditions:

(i) $a^2 \to 0$ as $\rho \to 0$, isentropically. This ensures that $dB_y/d\rho \to 0$ as $\rho \to 0$ along the slow branches.

(ii) $(\partial/\partial\rho)(\rho a^2)_s = \{a^2 + \rho(\partial^2 p/\partial\rho^2)_s\}$ always positive. This is a slightly stronger condition than the Weyl condition and requires that $(\partial^2 p/\partial\tau^2)_s$ be greater than a^2/τ^3. This condition is necessary to secure that only one singular point X occurs. The density at this point is given by the condition $\rho a^2 = B_x^2/\mu$ which fixes ρ if ρa^2 varies monotonically with ρ.

Other facts which permit the establishment of the form of the branches are as follows:

(*a*) Equation (9) is even in B_y, which implies symmetry in the ρ-axis.

(*b*) $dB_y/d\rho$ cannot change sign while B_y and ρ are finite and non-zero.

(*c*) From (7) and (8) we have $1-(B_y/\rho)\,d\rho/dB_y = b_x^2/c^2$, positive. Hence $|B_y/\rho|$ increases with ρ on the fast branches and B_y, b_y and c (fast) $\to \infty$ as $\rho \to \infty$, whereas $b_x \to 0$. Thus $(B_y/\rho)\,d\rho/dB_y \to 1$ and $B_y/\rho \to$ const. along the fast branches as $\rho \to \infty$. The fast waves are then tending towards the case treated earlier (Shercliff 1960 and others) where the field is essentially transverse, with $B_y \gg B_x$.

We observe that simple waves are either fast or slow throughout. The only exception to this occurs when $B_y = 0$ throughout, i.e. an ordinary simple wave for which $c = a$. When $B_y = 0$, $c = a$ or b_x. If $\rho < \rho_X$, $a < b_x$, and if $\rho > \rho_X$, $a > b_x$. In either case, the wave for which $dB_y/d\rho$ is infinite has $c = b_x$. The fact that the fluid enters or leaves a simple wave at a relative normal velocity of b_x where $B_y = 0$ makes an interesting contrast with the well-known result that the fluid enters or leaves a shock wave at a relative normal velocity of b_x when $B_y = 0$ *on the other side of the wave*, which is then a switch-on or switch-off shock.

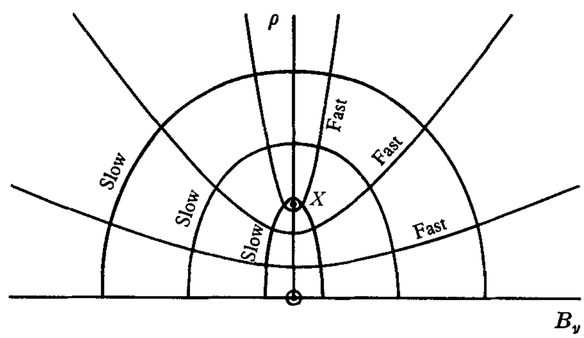

FIGURE 1. Trajectories on the B_y/ρ plane for simple waves.

In each slow wave there are upper limits on density and B_y, and in each fast wave a lower limit on density, except in the ordinary wave with $B_y = 0$.

If the pressure is negligible in comparison with the magnetic pressure, as in the ionosphere, we have the interesting degenerate case where $a = 0$. The slow and fast speeds then become zero and b, respectively. The fast waves still travel faster than the Alfvén speed b_x. Their trajectories on the B_y/ρ plane are the hyperbolae $B_y^2 + B_x^2 \propto \rho^2$, and along each trajectory $c \propto \rho^{\frac{1}{2}}$.

2.1. *Steepening of compressive simple waves*

From the preceding section we see that B_y cannot change sign in a wholly compressive wave and that a wave in which B_y does change sign must be partly expansive. But an expansive *and* compressive wave need not involve a change of sign of B_y unless the relevant extremum of ρ is reached.

It is natural to inquire whether compressive simple waves show the usual steepening tendency which leads to the creation of compressive shocks. At the same time expansive waves would show a spreading tendency. That this occurs has been stated by Liubarskii & Polovin (1958). The condition for its occurrence is that, for right-travelling waves,

$$\frac{d}{d\rho}(v_x + c) = \frac{c}{\rho} + \frac{dc}{d\rho}$$

be positive.

Differentiation of (6) and the use of (8) leads to the result

$$\frac{c}{\rho}+\frac{dc}{d\rho}=\frac{c}{2\rho}\frac{\left\{\tau^3\left(\frac{\partial^2 p}{\partial\tau^2}\right)_s+\frac{3(c^2-a^2)^2}{b_y^2}\right\}}{\left(a^2+\frac{(c^2-a^2)^2}{b_y^2}\right)}. \tag{10}$$

Granted Weyl's conditions, it follows from the form of (10) that $c/\rho+dc/d\rho$ is indeed positive and that fast or slow compressive waves, in which B_y does not change sign, do steepen, presumably into the fast or slow shocks that are discussed later. These too involve no change in sign of B_y. A simple wave in which B_y did change sign would not entirely steepen; the expansive parts would spread instead. Thus it is hard to see how the so-called intermediate shocks, in which B_y changes sign, could be created directly by a steepening process. This perhaps adds to the doubt which has been cast on the existence of intermediate shocks by Akhiezer *et al.* (1958) and Germain (1959) and others. Akhiezer *et al.* have shown that intermediate shocks, treated as discontinuities, are unstable to weak disturbances, while Germain has shown that intermediate shock structures cannot exist for certain values of the diffusivities. An obvious question deserving investigation is whether intermediate shocks could arise when a shock, newly formed from the compressive part of a simple wave, advances into an expansive part in which B_y is reversed.

In the degenerate cases where either the longitudinal or the transverse field component vanishes, condition (10) yields the results:

(i) If $B_x=0,\quad c^2=a^2+b^2$ and $\dfrac{c}{\rho}+\dfrac{dc}{d\rho}=\dfrac{\tau^4}{2c}\left\{\left(\dfrac{\partial^2 p}{\partial\tau^2}\right)_s+\dfrac{3B^2\rho^2}{\mu}\right\}$,

in agreement with the earlier paper (Shercliff 1960).

(ii) If $B_y=0$, *either*

$$c=a \quad\text{and}\quad \frac{c}{\rho}+\frac{dc}{d\rho}=\frac{\tau^4}{2a}\left(\frac{\partial^2 p}{\partial\tau^2}\right)_s,$$

the usual result for ordinary simple waves, *or*

$$c=b \quad\text{and}\quad \frac{c}{\rho}+\frac{dc}{d\rho}=\frac{3b}{2\rho}.$$

This result also applies to the case where $a=0$. In all these degenerate cases compressive waves still steepen.

3. Steady motions

In this section we generalize the theory of strictly one-dimensional steady motions (Shercliff 1960) to include the effect of a longitudinal field component. This involves our considering variation of transverse momentum. We continue to take the gas as having perfect electrical conductivity, except in § 3.1.

In steady one-dimensional gasdynamics some influence must be exerted on the flow in order that any changes should occur at all. By 'influence' is meant the application of additional forces to the flow (by viscous stress, distributed drag, actuator disks, gravity, etc.) or heat exchanges (by conduction, radiation or release or absorption of heat by chemical, nuclear, or thermonuclear reaction). It is possible to treat the resultant magnetogasdynamic processes in the symmetrical manner already developed for ordinary gasdynamics (Shercliff 1958). A difference here is that G, the mass flow per unit area, is constant because the motion is strictly one-dimensional.

Some of the processes may be difficult to achieve in reality, but nevertheless it can still be useful to consider them as mathematical loci since worthwhile results can be deduced from them, particularly about shocks. Germain (1959) goes further and considers mathematical loci where quantities such as G or the electric field E_z are varied. These cannot represent processes occurring in space, since G and E_z must be uniform by continuity and Maxwell's equations, as is pointed out below.

In steady motions, (1) degenerates to

$$\rho v_x = G \quad \text{(const.), assumed positive,}$$

and (2) gives

$$\rho v_x^2 + p + B_y^2/2\mu = F_x \quad \text{and} \quad \rho v_x v_y - B_x B_y/\mu = F_y,$$

where F_x and F_y are constant in the absence of additional forces or stresses in the x and y directions, respectively. Since $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$, we see that E_z is the only component of electric field, and Maxwell's equation $\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t = 0$ here indicates that E_z is uniform. Since B_x is also uniform and non-zero, we can suppress any E_z by the choice of axes moving in the y-direction with a velocity E_z/B_x. Then $\mathbf{v}\times\mathbf{B} = 0$ and we have the simplification that the flow and field are everywhere parallel. Hence $B_y/\rho v_y = B_x/G = \text{const.}$

Another consequence of the vanishing of $\mathbf{E}$ is that no electrical energy exchange $\mathbf{E}\cdot\mathbf{j}$ occurs, and the stagnation enthalpy, $H = h+\frac{1}{2}v^2 = h+\frac{1}{2}(v_x^2+v_y^2)$, is constant in the absence of any additional heat or work exchange.

The other major flow property which can be constant is the entropy s. This occurs in the absence of dissipation or heat exchange.

We thus have the situation where a flow characterized by known values of G and B_x may undergo changes in which some or all of the quantities F_x, F_y, H and s may be made to vary. For given values of G and B_x, *three* further quantities are necessary to specify the state of a flow-section (e.g. ρ, s and v_y). Another set of three quantities which may be used as co-ordinates is (F_x, F_y, H), but then the flow-state is not uniquely defined. We shall see that there may be up to four states having these co-ordinates, mutually accessible via stationary shocks, since across a stationary shock F_x, F_y and H are conserved (de Hoffman & Teller 1950). Note that across shocks G and B_x are also uniform and $\mathbf{v}$ can be made parallel to $\mathbf{B}$ on both sides. The fact that only three of the four quantities F_x, F_y, H and s are independent is expressible by the relation

$$v_x dF_x + v_y dF_y + GT\,ds = G\,dH, \tag{11}$$

which results from the relation $T\,ds = dh - dp/\rho$ for a gas in equilibrium.

We shall concentrate on the six processes which may be defined by keeping any two of F_x, F_y, H and s constant. The processes may be interpreted physically in various ways. The most interesting case is the generalized Rayleigh process, in which F_x and F_y are conserved because no extra force is applied to the fluid, but H and s are changed by heat exchanges. Similarly we have a generalized Fanno process, in which F_y and H are constant while dissipative forces operate in the x-direction but no energy is exchanged. Another adiabatic case is that where gravity or other forces in the x-direction result in reversible energy exchange and F_y and s are constant. Alternatively there may be forces in the y-direction, causing F_y to vary.

Some simple, important relations are

$$(\tau-\tau_0)\,B_y = F_y B_x/G^2 \quad\text{and}\quad (\rho_0-\rho)\,v_y = \mu F_y G/B_x^2, \tag{12}$$

where

$$\tau_0 = B_x^2/\mu G^2 = \text{const.},$$

and

$$\tau/\tau_0 = v^2/b^2 = v_x^2/b_x^2 = v_y^2/b_y^2. \tag{13}$$

Relation (12) is particularly important when $F_y = \text{const.}$, as obviously the specific volume τ_0 is a singular value. Its importance has also been noted by Germain (1959). When $\tau = \tau_0$, $v_x = b_x$, i.e. the flow is travelling at the Alfvén speed.

In all six processes, the two of F_x, F_y, H and s which are not being kept constant are stationary whenever v_x equals the local fast or slow sound speed given by (6). This is easily demonstrated, for instance, in the case (H, s) const. We have

$$\left(\frac{\partial F_x}{\partial \rho}\right)_{Hs} = a^2 - v_x^2 + \frac{B_y}{\mu}\frac{dB_y}{d\rho}, \tag{14}$$

$$\left(\frac{\partial H}{\partial \rho}\right)_{Hs} = 0 = \frac{a^2}{\rho} - \frac{v_x^2}{\rho} + v_y\frac{dv_y}{d\rho},$$

and

$$\frac{dB_y}{B_y} = \frac{d\rho}{\rho} + \frac{dv_y}{v_y},$$

which together with (13) yield

$$\left(\frac{\partial F_x}{\partial \rho}\right)_{Hs} = -\frac{1}{v_x^2}\{v_x^4 - v_x^2(a^2+b^2) + a^2 b_x^2\} = -\frac{X}{v_x^2}, \quad \text{say},$$

where X vanishes whenever v_x equals either root of (6).

The values of all such derivatives are presented in table 1 as multiples of X. The quantity Y in the last line denotes the expression

$$\left\{1 - \left(1 - \frac{b_x^2}{v_x^2}\right)\frac{1}{T}\left(\frac{\partial h}{\partial s}\right)_\rho\right\}.$$

Since the gas has a positive expansion coefficient, $(\partial s/\partial p)_\rho$ and $(\partial s/\partial h)_\rho$ in table 1 are positive and $T(\partial s/\partial h)_\rho$ is less than unity. As a result Y can vanish, for example for a perfect gas when $\tau/\tau_0 = \gamma/(\gamma-1)$. We shall not investigate this complicated (F_x, H) case any further, despite the fact that it is one of the processes that link the two sides of shocks.

Not all parts of these processes may be physically meaningful. In the Rayleigh (F_x, F_y) process, B_y increases so much as $\tau \to \tau_0$ that $B_y^2/2\mu$ becomes larger than F_x and the pressure goes negative. Similarly, in the (F_x, s) process, B_y^2 and v_y^2 become negative whenever p or ρv_x^2 gets too large, at low and high values of τ, respectively.

In the processes with F_y constant the derivatives in table 1 increase without limit as τ passes τ_0, and v_y and B_y change sign via large values.

When F_y is not constant, the signs of F_y, v_y and B_y are to some extent arbitrary, because only the squares of v_y and B_y appear in F_x and H. We can either keep F_y constant in sign so that v_y and B_y change sign discontinuously as τ passes τ_0, although F_x and H would be continuous (this course would be appropriate if we were interested in states having the same value of F_y, for example, the two sides of a shock) or we can keep v_y and B_y (and b_y) positive and continuous, for convenience in discussing table 1. Then F_y changes sign via zero as τ passes τ_0.

Const.	$\partial F_x/\partial\rho$	$\partial F_y/\partial\rho$	$\partial H/\partial\rho$	$\partial s/\partial\rho$
H, s	$-\dfrac{1}{v_x^2}$	$\dfrac{1}{v_x v_y}$	0	0
F_x, s	0	$\dfrac{1}{b_x b_y}$	$\dfrac{1}{\rho b_x^2}$	0
F_y, s	$-\dfrac{1}{v_x^2-b_x^2}$	0	$-\dfrac{1}{\rho(v_x^2-b_x^2)}$	0
F_x, F_y	0	0	$T\left(\dfrac{\partial s}{\partial p}\right)_\rho \dfrac{1}{(v_x^2-b_x^2)}$	$\left(\dfrac{\partial s}{\partial p}\right)_\rho \dfrac{1}{(v_x^2-b_x^2)}$
F_y, H	$-T\left(\dfrac{\partial s}{\partial h}\right)_\rho \dfrac{1}{(v_x^2-b_x^2)}$	0	0	$\dfrac{1}{\rho}\left(\dfrac{\partial s}{\partial h}\right)_\rho \dfrac{1}{(v_x^2-b_x^2)}$
F_x, H	0	$\dfrac{1}{v_x v_y Y}$	0	$-\dfrac{1}{T\rho v_x^2 Y}$

TABLE 1. Derivatives expressed as multiples of X.

The next task is to establish whether the stationary values of F_x, F_y, H and s are maxima or minima. From relations such as

$$\left(\frac{\partial X}{\partial\rho}\right)_{F_y H} = \left(\frac{\partial X}{\partial\rho}\right)_{F_y s} + \left(\frac{\partial X}{\partial s}\right)_{\rho F_y}\left(\frac{\partial s}{\partial\rho}\right)_{F_y H},$$

we see that $\partial X/\partial\rho$ takes the same value in all the processes when $X = 0$. The quantities such as $(\partial X/\partial s)_{\rho F_y}$ which enter are finite for all likely gases. In fact

$$(\partial X/\partial s)_{\rho F_y} = (b_x^2 - v_x^2)\,\tau^2(\partial^2 T/\partial\tau^2)_s.$$

The common value of $\partial X/\partial\rho$ is most easily evaluated in the (H, s) case, using (7) with $v_x = c$ and the result

$$\frac{\partial b^2}{\partial\rho} = -\frac{b^2}{\rho} + \frac{2B_y}{\mu\rho}\frac{\partial B_y}{\partial\rho} = -\frac{b^2 + 2(v_x^2 - a^2)}{\rho} \quad \text{from (14) when } X = 0.$$

We find that
$$\frac{\partial X}{\partial\rho} = \frac{(b_x^2 - v_x^2)}{\rho}\left\{\tau^3\left(\frac{\partial^2 p}{\partial\tau^2}\right)_s + \frac{3(v_x^2 - a^2)^2}{b_y^2}\right\},$$

an expression reminiscent of the denominator of (10). We conclude that in each case $\partial X/\partial\rho$ has the same sign as $(b_x^2 - v_x^2)$ when $X = 0$. For slow roots of $X = 0$,

$(b_x^2 - v_x^2)$ and $(\rho - \rho_0)$ are positive, for fast, negative. We can thus compile table 2, with the convention that b_y, B_y and v_y are positive in the first two processes. They can be completed by changing the sign of b_y, B_y, v_y and F_y throughout.

It is easily verified that, in each process, the state is a single-valued function of ρ (apart from the ambiguity of sign of F_y already mentioned), provided the state of the gas is uniquely defined by ρ and any one of s, p or h. Each process will have only one sonic point of each kind, the fast when $\rho < \rho_0$, the slow when $\rho > \rho_0$. This is illustrated in figure 2. Extra sonic points such as P and Q are precluded because on each continuous branch, maxima and minima must occur alternately but the nature of the extrema are stated in table 2. Figure 2 shows the difference between cases such as (*a*) where $F_y =$ const. and (*b*), (*c*) and (*d*) where F_y varies. In the latter case the ambiguity of sign of F_y, already discussed, is shown by the curves (*c*) and (*d*).

Const.	Slow sonic point	Fast sonic point
H, s	F_x max.; F_y min.	F_x min.; F_y max.
F_x, s	F_y min.; H min.	F_y max.; H max.
F_y, s	F_x min.; H min.	
F_x, F_y	H max.; s max.	
F_y, H	F_x min.; s max.	

TABLE 2

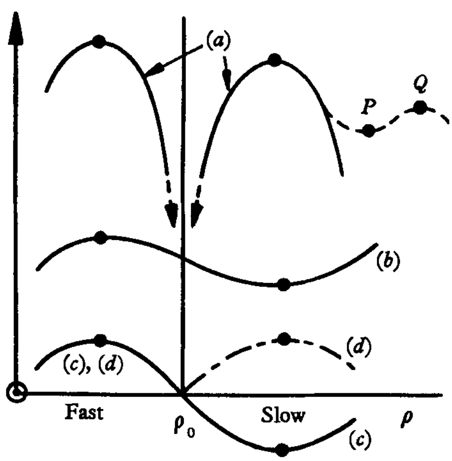

FIGURE 2. Typical graphs of the five processes. (*a*) H or s against ρ with (F_x, F_y) const. (Rayleigh line). (*b*) H against ρ with (F_x, s) const. (*c*) F_y against ρ with (F_x, s) const., B_y and v_y positive. (*d*) F_y against ρ with (F_x, s) const., F_y positive.

As in the earlier papers (Shercliff 1958, 1960) it is possible to explore the significance of the isothermal sound speeds, determined by (6) with a^2/γ replacing a^2. The most important result which emerges is that the temperature T is a maximum at the two isothermal-sonic points that occur in a Rayleigh process, which also straddle the point $\rho = \rho_0$. This requires the further assumption that $(\partial^2 p/\partial \tau^2)_T$ be positive, which is true for gases except near the critical point. As with ordinary shocks, the Rayleigh process represents the shock structure in gases for which the thermal diffusivity is dominant. The isothermal fast and slow sound speeds are then critical for determining whether or not there is a shock-within-a-shock, in which the other diffusivities operate.

The properties of Rayleigh lines are more thoroughly exploited in the study of shock transitions in § 4.

3.1. *Ordinary generalized gas dynamics*

A degenerate case of the foregoing is worth studying briefly. This is the case which results from setting B_x and B_y to zero and which represents ordinary one-dimensional gas dynamics generalized to include transverse effects, but with the flow per unit area kept constant (although it is simple to waive this restriction). Again influences must be exerted on the fluid to cause changes to take place. These may still be electromagnetic in origin, just as electromagnetic effects were considered in the earlier work on ordinary gasdynamics (Shercliff 1958). As an example we can consider a flow at such low magnetic Reynolds number that the field may be taken to be uniform but at any inclination. Alternatively, we may imagine the processes as being somewhat idealized versions of the flows that occur in axial-flow turbo-machinery.

Adopting the previous techniques, we consider the quantities G (here constant), H and s, as usual, together with $f_x = p + \rho v_x^2$ and $f_y = \rho v_x v_y$. They are related by

$$v_x df_x + v_y df_y = G\,dH - GT\,ds. \tag{15}$$

We could generalize to three dimensions with a term $v_z df_z$. Again processes can be defined by taking two of H, s, f_x and f_y constant. Except in one case, the other two variables are then stationary when $v_x = a$ and the longitudinal Mach number is unity.

The six processes are:

f_x, s const. Here the state of the gas and v_x are constant and only v_y, f_y and H vary. This corresponds to flow through impulse blading. The condition $v_x = a$ is irrelevant.

f_y const. and one of s, f_x or H const. There are merely the three processes (G, s), (F, G) or (G, H) treated in the earlier paper (Shercliff 1958) with a constant transverse velocity v_y superposed. F and f_x are identical, but H in the earlier paper did not include the term $\frac{1}{2}v_y^2$. Now the condition $v_x = a$ is critical.

f_x, H const. This is a modification of the (F, G) Rayleigh process, in which p, ρ, s and v_x vary as usual, but in addition v_y varies in such a way that H is constant. The condition $v_x = a$ makes s and $(h + \frac{1}{2}v_x^2)$ a maximum and hence v_y and f_y a minimum. Indeed if H is too small, v_y will fall to zero before v_x reaches a. One way of realizing this process would be to have a uniform field in the longitudinal direction, the magnetic Reynolds number being low, and zero electric field. Then in this adiabatic, dissipative process the velocity v_y across the field would decay to zero or until choking occurred at $v_x = a$.

H, s const. This is a modification of the (G, s) process discussed in the earlier paper, in which p, ρ, s and v_x vary as usual, but v_y varies in such a way that H is constant. The condition $v_x = a$ makes f_x and $(h + \frac{1}{2}v_x^2)$ a minimum and hence

v_y and f_y a maximum. Instead of the external energy exchange in the original (G, s) process, the transverse kinetic energy now acts as a sink or source of energy. It is interesting to observe that this transverse energy is a maximum when the *longitudinal* velocity is sonic. The process can be imagined to be brought about by stationary cascades of blades which apply forces to the fluid in a direction perpendicular to its motion, so that

$$v_x df_x + v_y df_y = 0,$$

in accordance with (15). It is perhaps necessary to point out that turbine nozzle rows are not subject to an upper limit on v_y when $v_x = a$, because G is not usually constant in them.

The list of processes could be prolonged by putting other constraints on f_x, f_y, H and s. An example is the flow in zero electric field and a uniform oblique magnetic field (at low magnetic Reynolds number) where the process is defined by specifying df_x/df_y (which equals $-B_y/B_x$) and $H = \text{const.}$

An important member of this family of processes is that which occurs as the structure of a shock in which the magnetic diffusivity is dominant, as considered by Ludford (1959), Bleviss (1959) and Germain (1959). Here we resurrect Maxwell's equations and take $\mathbf{E}$ as zero. Since the conductivity is finite the process does not belong to the family of flows with $\mathbf{v}$ and $\mathbf{B}$ parallel, already discussed. In particular the extrema of entropy, etc., occur when $v_x = a$, the ordinary sound speed, not the fast or slow ones. Thus a is the critical velocity for determining whether or not there is a shock-within-a-shock, in which the other diffusivities operate. Here the process is defined by taking $H = \text{const.}$ and by relating f_x and f_y via the parameter B_y through the equations

$$f_x + B_y^2/2\mu = \text{const.}, \quad f_y - B_x B_y/\mu = \text{const.}$$

It is not proposed to discuss this process further here as we are not primarily concerned with shock structure.

4. Shock transitions

Ever since the Rankine–Hugoniot relations were generalized to include magnetogasdynamic effects by de Hoffman & Teller (1950), the nature of their solutions has been studied by many authors, notably Lüst (1953) and Friedrichs (1954). The results of these and other writers are discussed at the end of this section.

More recently, Germain (1959) has investigated the properties of possible shock transitions. His work is very close to the work reported here in many respects. He too considers an arbitrary gas obeying Weyl's conditions, and also uses the generalized Rayleigh line to establish a classification of shocks by relating the velocity normal to the shock to the slow, fast and Alfvén wave speeds. However, his proof that all shocks must be compressive involves two further loci, one a generalization of the Hugoniot curve, also used by Friedrichs & Kranzer (1958) and Iordanskii (1958), the other a complicated locus somewhat comparable to the (F_x, H) process mentioned earlier. The use of this locus leads him to postulate an extra, unnecessary restriction on the gas properties.

In this paper it is shown how nearly all the results follow simply from the properties of the generalized Rayleigh line. For completeness we start from first principles, taking the x-axis normal to the plane of the shock, i.e. in the direction of one-dimensional variation, with the shock stationary in the co-ordinate system. Maxwell's equations, $\operatorname{div}\mathbf{B} = 0$ and $\operatorname{curl}\mathbf{E} = 0$, then indicate that B_x, E_y and E_z are constant. Since B_x is non-zero we can choose axes moving in the y- and z-directions so as to make E_y and E_z vanish. The shock layer separates two uniform regions devoid of current, since $\operatorname{curl}\mathbf{B} = 0$ in them. Thus Ohm's law indicates that $(\mathbf{v}\times\mathbf{B})_y$ and $(\mathbf{v}\times\mathbf{B})_z$ vanish outside the shock, and $v_z/B_z = v_x/B_x = v_y/B_y$. The shock layer contains a current sheet perpendicular to and responsible for the discontinuity in $\mathbf{B}$ across the shock. We choose the z-axis parallel to this current sheet so that $B_{z1} = B_{z2} = B_z$, say, if we use the suffices 1 and 2 to denote the two sides of the shock. In the shock the fluid suffers a net magnetic force perpendicular to the current sheet. Thus z-momentum is conserved and $v_{z1} = v_{z2} = v_z$, say. Hence $v_z/v_{x1} = B_z/B_x = v_z/v_{x2}$, and *either* (i) $v_{x1} = v_{x2} = v_x$, say, *or* (ii) $v_z = B_z = 0$, two distinct cases.

4.1. *Case* (i). *The Alfvén shock* $(v_{x1} = v_{x2} = v_x)$

The equation of continuity,

$$G_1 = G_2, \quad \text{where} \quad G = \rho v_x, \tag{16}$$

here indicates that $\rho_1 = \rho_2 = \rho$, say. The y-momentum equation is

$$F_{y1} = F_{y2}, \quad \text{where} \quad F_y = Gv_y - B_x B_y/\mu, \tag{17}$$

which here leads to

$$\rho v_x(v_{y1} - v_{y2}) = B_x^2(v_{y1} - v_{y2})/\mu v_x \quad \text{since} \quad v_{y1}/B_{y1} = v_x/B_x = v_{y2}/B_{y2}.$$

It follows that $v_x = \pm B_x/(\mu\rho)^{\frac{1}{2}}$ if we reject the trivial case $v_{y1} = v_{y2}$. The wave is seen to be an Alfvén shock, also called a transverse or symmetrical shock by some. The energy equation is

$$H_1 = H_2, \quad \text{where} \quad H = h + \tfrac{1}{2}v^2 \quad (v^2 = v_x^2 + v_y^2), \tag{18}$$

since $\mathbf{E}\,.\,\mathbf{j}$ is zero everywhere. For the Alfvén shock this gives

$$h_1 + B_{1y}^2/2\mu\rho = h_2 + B_{2y}^2/2\mu\rho.$$

The x-momentum equation is

$$F_{x1} = F_{x2}, \quad \text{where} \quad F_x = p + Gv_x + B_y^2/2\mu, \tag{19}$$

which here yields $p_1 + B_{1y}^2/2\mu = p_2 + B_{2y}^2/2\mu$. We see that $u_1 = u_2$, where u is the internal energy $(h - p/\rho)$, and that the thermodynamic state, uniquely defined by u and ρ, is unchanged by the shock. It follows that $s_1 = s_2$ and $|\mathbf{B}_1| = |\mathbf{B}_2|$. The transverse field component simply rotates arbitrarily without changing in magnitude. Alfvén shocks in which it rotates through 180° (i.e. $B_z = 0$) also belong to Case (ii).

4.2. *Case* (ii). *Two-dimensional shocks* ($B_z = 0$)

Here the initial and final fields lie in a plane perpendicular to the shock. The array of solutions to the conservation equations (16) to (19) together with

$$B_x = (B_y v_x / v_y)_{1 \text{ or } 2} \tag{20}$$

is much more complicated than in Case (i). We shall approach the problem by exploiting the fact that the states on each side of the shock lie on a Rayleigh line, a locus defined by keeping B_x, G, F_x and F_y constant, subject to condition (20). The Rayleigh line is most simply specified in terms of p and τ (the specific volume $1/\rho$). The resulting relation is

$$F_x = p + G^2\tau + F_y^2 B_x^2 / 2\mu G^4 (\tau - \tau_0)^2, \tag{21}$$

where $\tau_0 = B_x^2/\mu G^2 =$ const. It is instructive to consider the family of Rayleigh lines characterized by fixed values of F_x, B_x, G and τ_0 and a series of values of $|F_y|$. These obviously take the form illustrated in figure 3, as Germain (1959) has

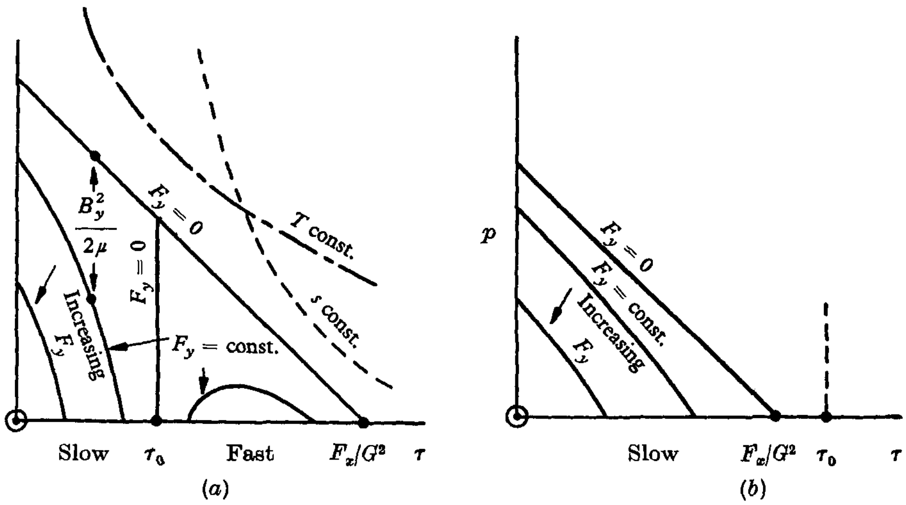

FIGURE 3. Rayleigh lines for given (G, B_x, F_x) and various F_y on the p-τ plane. (a) $F_x > B_x^2/\mu$, (b) $F_x < B_x^2/\mu$.

remarked. The condition for the occurrence of two branches (figure 3*a*) is $F_x > B_x^2/\mu$. The branches are labelled fast and slow according as $v_x >$ or $< b_x$, where $b_x = B_x/(\mu\rho)^{\frac{1}{2}}$, the longitudinal Alfvén speed. In fact $v_x^2/b_x^2 = \tau/\tau_0$. Also

$$B_y(\tau - \tau_0) = F_y B_x / G^2 \quad \text{and} \quad v_y(\tau - \tau_0) = \tau F_y / G. \tag{22}$$

Hence the line $F_y = 0$ consists of two straight lines, the ordinary Rayleigh line $F_x = p + G^2\tau$, ($B_y = 0$), and the part of the vertical line $\tau = \tau_0$ that lies below it (since $B_y^2/2\mu$ is positive). In physical terms the latter could represent a kind of Alfvén wave in which changes are brought about by heat exchange or heat release. As $|F_y|$ increases from zero in figure 3*a*, the fast branch disappears when $F_y^2 \geqslant 8\mu(F_x - B_x^2/\mu)^3/27B_x^2$ and the slow branch disappears when $F_y^2 \geqslant 2B_x^2 F_x/\mu$.

The fast disappears first if $4B_x^2/\mu > F_x$, and vice versa. These inequalities should be regarded as limitations on the shock states possible for given values of F_x, F_y and B_x.

Figure 3 also displays the dependence of B_y on τ along a Rayleigh line since the vertical intercept between that line and the ordinary Rayleigh line equals $B_y^2/2\mu$. For a given value of F_y, B_y and v_y take different signs in the slow and fast regions, because of (22).

A vital fact about Rayleigh lines is that along them $T\,ds = dH$. This is easily verified from (16) to (20) using $T\,ds = dh - dp/\rho$. We are interested in states on Rayleigh lines having the same values of H so as to be mutually accessible via shocks and having increasing entropy as one proceeds downstream. This leads us to follow the standard procedure of considering the lines on the T-s diagram.

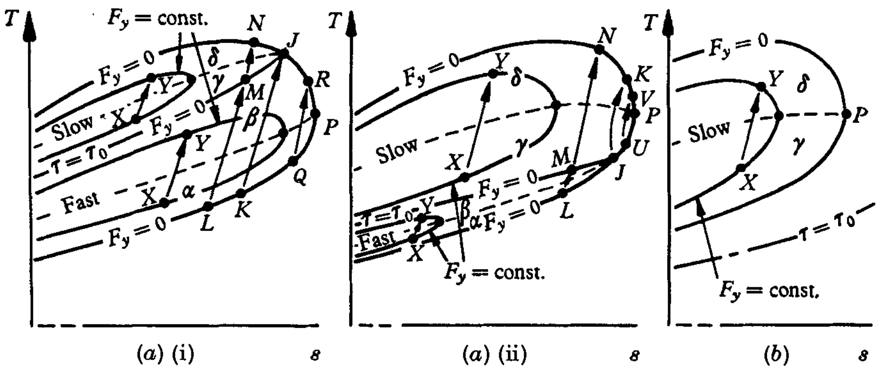

FIGURE 4. Rayleigh lines for given (G, B_x, F_x) and various F_y on the T-s plane. (*a*) $F_x > B_x^2/\mu$, (i) $\tau^* > \tau_0$, (ii) $\tau^* < \tau_0$. (*b*) $F_x < B_x^2/\mu$.

Since derivatives such as $(\partial p/\partial \tau)_s$, $(\partial p/\partial \tau)_T$, $(\partial p/\partial s)_\tau$, etc., have been assumed constant in sign, the p-τ and T-s diagrams are very simply related topologically. Since also $(\partial^2 p/\partial \tau^2)_s$, $(\partial^2 p/\partial \tau^2)_T$ are taken positive, Rayleigh lines and lines of constant s or T have opposite curvature in the p-τ diagram, and points on Rayleigh lines where s or T are stationary are clearly *maxima* of s or T. Moreover, only one point of each kind occurs on each (slow or fast) branch of a Rayleigh line. It follows that the lines for given (F_x, B_x, G) must be disposed as shown in figure 4. More rigorous proofs of the properties of the Rayleigh line are available in § 3.

The line $F_y = 0$ consists of the familiar ordinary Rayleigh line, with its entropy maximum at P where $v_x = a$ and $\tau = \tau^*$, say, plus part of the constant volume line $\tau = \tau_0$. The distinction between (*a*) (i) and (*a*) (ii) lies in the relative positions of P and the point J where $v_x = b_x$ and $\tau = \tau_0$. For case (*a*) (i) we need $\tau^* > \tau_0$. This is a condition on F_x, B_x and G. To be more specific one needs more facts about the gas; if it is a perfect gas then the condition for (*a*) (i) is $F_x > (\gamma+1)B_x^2/\gamma\mu$.

As $|F_y|$ increases from zero the branches move downwards and leftwards in figure 4 and finally disappear in accordance with the inequalities already discussed. Note that figures 3 and 4 are independent of the sign of F_y, since F_y appears squared in (21).

4.2.1. *Fast and slow shocks*

Since $T\,ds = dH$ along Rayleigh lines, to a point such as X on each branch there corresponds another point Y on the same branch having the same value of H and placed more or less as shown in figure 4 to satisfy the condition $\int_X^Y T\,ds = 0$. Such pairs of points satisfy the conservation relations for a shock. Moreover, X, the point of lower entropy (and lower density in view of figure 3) must be upstream. Thus shocks of this type, which we shall call slow or fast, depending on which branch they belong to, must be compressive and longitudinally decelerative. From (22) it follows that $|B_y|$ and $|v_y|$ increase across fast shocks and decrease across slow shocks. Across both types B_y and v_y do not change sign. We observe that B_y varies in the same way as in the slow and fast compressive simple waves discussed in § 2.

As we increase H, and X moves rightwards along its branch of a Rayleigh line, X and Y finally coincide at the entropy maximum, also a maximum of H. In this limit the shock becomes a weak sound wave across which F_x, F_y, H and s are conserved and at which v_x equals either c_f, the fast, or c_s, the slow sound speed given by the well-known equation

$$c^4 - c^2(a^2+b^2) + a^2 b_x^2 \equiv (c^2 - c_s^2)(c^2 - c_f^2) = 0. \tag{23}$$

The dotted lines in figure 4 are the loci of these sonic points as F_y varies, which together with the line $\tau = \tau_0$ divide the ordinary Rayleigh loop into four zones α, β, γ, δ (two in figure 4*b*). From (21) and $ds/d\rho = (\partial s/\partial p)_\rho\,(dp/d\rho - a^2)$, or otherwise, it follows that along a Rayleigh line,

$$\frac{ds}{d\rho} = \left(\frac{\partial s}{\partial p}\right)_\rho \frac{(v_x^2 - c_s^2)(v_x^2 - c_f^2)}{(v_x^2 - b_x^2)},$$

in which c_s, c_f and b_x are *local* values. Note that $c_f^2 \geqslant b_x^2 \geqslant c_s^2$ and $(\partial s/\partial p)_\rho$ is positive. It is evident from figures 3 and 4 that $ds/d\rho$ is positive in regions α and γ, negative in β and δ, while $(v_x^2 - b_x^2)$ is positive in α and β, negative in γ and δ. Since $b_x^2 \geqslant c_s^2$, $(v_x^2 - c_s^2)$ is also positive in α and therefore $(v_x^2 - c_f^2)$ is positive there too. In this fashion we may readily establish the following inequalities that characterize states lying in the four zones:

$$\left.\begin{array}{ll}(\alpha) & v_x > c_f,\\ (\beta) & c_f > v_x > b_x,\\ (\gamma) & b_x > v_x > c_s,\\ (\delta) & c_s > v_x,\end{array}\right\} \tag{24}$$

if we take all these velocities as positive.

In particular we observe that the normal velocity v_x in slow and fast shocks jumps from supersonic to subsonic relative to the slow and fast sound speeds, but stays respectively below or above the Alfvén speed.

4.2.2. *Switch-on, switch-off, Alfvén and ordinary shocks*

The case $F_y = 0$ deserves closer scrutiny, since here the upper arm of the fast branch and lower arm of the slow branch have coalesced. Typical fast shocks now appear in figure 4 as LM or QR and typical slow shocks as MN or UV. The shock LN also satisfies the conservation and entropy conditions. Shocks QR, UV and LN are all ordinary shocks since the end points all lie on the ordinary Rayleigh line where $B_y = 0$ and B_x is ineffectual.

At M, $v_x = b_x$ and $B_y \neq 0$. Thus LM is what Friedrichs & Kranzer (1958) call a switch-on shock, while MN is the switch-off shock. They are extreme cases of fast and slow shocks, respectively. On the side where the field is oblique the velocity equals the Alfvén velocity; on the side where the field is normal to the shock, $c_f =$ the greater of a and b_x, $c_s =$ the lesser, and (24) indicate that upstream of a switch-on shock $v_x >$ both a and b_x, while downstream of a switch-off shock $v_x <$ both a and b_x.

The point M represents two states, differing only in the sign of B_y and v_y. This point therefore represents an Alfvén shock (in the case $B_z = 0$) in which B_y and v_y reverse but the thermodynamic state (T, s) is unchanged.

Switch-on and off shocks are not possible for all values of F_x, G, B_x and H, even granted that $F_x > B_x^2/\mu$, to permit the $\tau = \tau_0$ branch. Given values of F_x, G and B_x fix the point J in figure 4. Switch-on or off shocks will obviously be possible only for smaller values of H than its value at J. In other words, a switch-on shock is not possible starting from a state from which an ordinary shock leads to a state at which $v_x > b_x$, i.e. starting from points to the right of K in figure 4(*a*)(i), K being the other point having the same H as J. Equally it is not possible to jump to points to the right of K in figure 4 (*a*)(ii) via switch-off shocks.

As H increases to its limit for switch-on shocks, the related switch-off shock weakens to vanishing point, and vice versa (e.g. M and $N \to J$ as $L \to K$). These limits on switch-on and off shocks are expressible in many ways. Bleviss (1959) in effect gives them as upper and lower limits on B_x for a given upstream state, L say, where $B_y = 0$. Then B_x and τ_0 must be such that J lies between L and N.

4.2.3. *Intermediate shocks*

We now consider shocks which join points lying on *different* branches of a Rayleigh line, in which therefore B_y, v_y and $(v_x^2 - b_x^2)$ change sign. Following Germain (1959) we call these intermediate shocks. Provided F_x, F_y and B_x satisfy the conditions for two branches to exist, there will be two pairs of states lying on the two different branches and having the same value of H, provided H is less than the lesser of the two maxima of H for given G, B_x, F_x and F_y, which may lie on either the slow or the fast branch. That there can be up to four states mutually accessible via shocks was observed by Ludford (1959) in the case of a perfect gas. We shall number these four states, 1, 2, 3 and 4 in order of increasing density, 1 and 2 being fast, 3 and 4 being slow. The fast shock 12 and slow shock 34 have already been discussed. There will also be four intermediate ones which travel in a direction to be determined from the entropy condition.

The four states, 1, 2, 3 and 4 lie respectively in the regions α, β, γ and δ and the inequalities (24) apply to them.

Consider the two states 2 and 3 on the two arms closest to the $\tau = \tau_0$ line. Suppose $s_2 \geqslant s_3$, as shown in figure 5. Consider the isentropics through 2 and 3 which cut the $F_y = 0$ Rayleigh line in W and Z. Figure 5 shows the two possibilities. From (16) to (20), with G, B_x, F_x and s constant, we find that

$$\partial H/\partial \rho = (v_x^2 - c_s^2)(v_x^2 - c_f^2)/\rho b_x^2,$$

which is negative along $2W$ and $Z3$, in the β and γ zones. Along $2W$ and $Z3$, ρ increases monotonically since $(\partial T/\partial \rho)_s$ is positive. Thus H decreases along

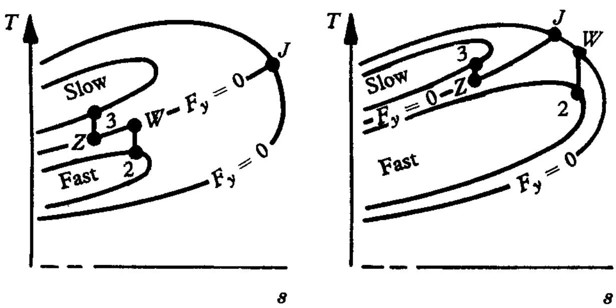

FIGURE 5

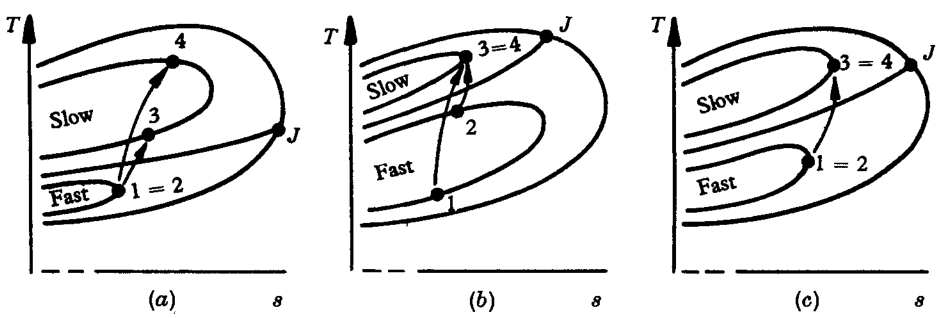

FIGURE 6. Rayleigh lines on the T-s plane, showing shocks that are (a) sonic ahead, (b) sonic behind and (c) sonic ahead and behind.

$2W$ and $Z3$, as it does along WZ since $T\,ds = dH$. We conclude, wrongly, that $H_2 > H_3$. Inevitably, then, $s_2 < s_3$. In fact the states are numbered in order of increasing entropy, and we conclude that all six shocks joining them must involve increases of density and hence of pressure, since $(\partial p/\partial \rho)_s$ and $(\partial p/\partial s)_\rho$ are both positive and non-zero, except at extreme states, such as those for a perfect gas where T, a^2 and p all tend to zero. (For a perfect gas we have $(\partial p/\partial s)_\rho = (\gamma - 1)p/R$.) Moreover, except at such states, we see that h also increases across shocks since $dh = T\,ds + \tau\,dp$, and hence that the kinetic energy $\frac{1}{2}(v_x^2 + v_y^2)$ decreases, H being constant.

In all intermediate shocks, upstream the velocity is above the local Alfvén velocity, downstream it is below it. The four types 13, 14, 23 and 24 are distinguished by the inequalities (24) applied appropriately upstream and downstream.

Intermediate shocks may be sonic ahead or behind; this occurs when H equals the lesser maximum of H for given values of G, B_x, F_x and F_y, and is illustrated in figure 6. If G, B_x, F_x and F_y are suitably matched, the two maxima of H are equal and the strangest case of all, sonic ahead and behind, can occur (figure 6*c*). If H is increased beyond the lesser of the two maxima for given G, B_x, F_x and F_y, intermediate shocks no longer occur.

The Alfvén shock is the limit of 23 shocks as $F_y \to 0$. Similarly 13, 14 and 24 shocks in this limit become switch-on, ordinary and switch-off shocks, respectively.

4.2.4. *Strong shocks*

Questions so far avoided are whether H has lower limits in addition to upper limits on a Rayleigh line, whether H can be so low that members of the set of states 1, 2, 3, 4 disappear and whether all parts of a Rayleigh line are accessible via shocks. In ordinary gasdynamics the high density end of the Rayleigh line is not accessible from shocks, since for very strong shocks having infinite pressure ratio the density ratio is finite.

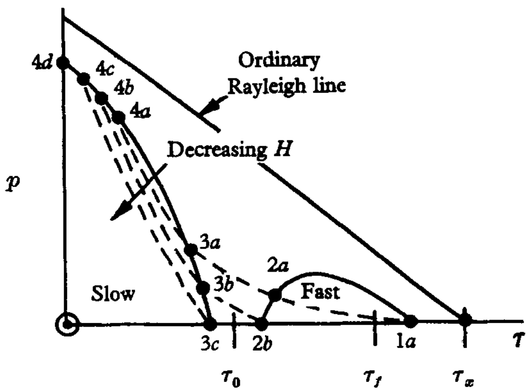

FIGURE 7. Rayleigh line on the p-τ plane showing strong shocks. $\tau_0 = B_x^2/\mu G^2$, $\tau_x = F_x/G^2$, $\tau_f = \frac{1}{3}(\tau_0 + 2\tau_x)$.

Figure 7 shows a two-branched Rayleigh line, cutting the p and τ axes in the points 1*a*, 2*b*, 3*c* and 4*d*. It may be shown that, whatever the behaviour of the gas, the kinetic energy $\frac{1}{2}(v_x^2 + v_y^2)$ is successively less at these four points in the above order. The proof of this simple result is to be found in the appendix. To make progress we must plausibly assume that at these states at very low p (or τ at 4*d*) the enthalpies h may be taken as equal. This is valid, for instance, if the gas approaches the perfect gas state there. It implies that the stagnation enthalpy H is successively less at the four points in order, so that at the point 2*a*, at which H has the *same* value as at 1*a*, H has a higher value than at 2*b* and so 2*a* lies off the zero pressure axis. [N.B. $dH/d\tau$ and dH/dp are finite at 2*b* in view of table 1 provided $T(\partial s/\partial p)_\rho$ is finite there. This is certainly true for a perfect gas, for which $T(\partial s/\partial p)_\rho = \tau/(\gamma-1)$.] The shock 1*a*2*a* is a fast *strong* shock with an infinite pressure ratio but a finite density ratio. In a similar way, we can deduce the occurrence of the five other strong shocks 1*a*3*a*, 1*a*4*a*, 2*b*3*b*, 2*b*4*b*, 3*c*4*c*. In figure 7 the dotted lines loosely connect points at which H is the same.

Of course it is possible that the lines of higher H may not intersect the slow branch at all and some of the six strong shocks may be missing. If the Rayleigh line has only one branch, only one strong shock occurs. All the strong shocks have a finite density ratio.

Several parts of the Rayleigh line are inaccessible to shocks. No shock can lead to the arcs $2a2b$, $3b3c$ or $4c4d$, though shocks can start from the first two of these.

Finally, we remark that, for a given Rayleigh line specified by (G, B_x, F_x, F_y), as H is lowered the states 1, 2, 3 and 4 will disappear in that order as the pressure at each state falls to zero in turn. In other words, members of the family of six shocks characterized by given values of (G, B_x, F_x, F_y, H) may be missing because H is too low, just as they may be missing because H is too high and for other reasons already discussed.

4.3. *Comparison with previous work on shocks*

Our simple classification of shocks into six types obviously needs reconciling with previous classifications. Germain's (1959) work has already been mentioned. Friedrichs's earlier work (1954) had only limited circulation, but one gathers he classified shocks as fast, Alfvén and slow, and omitted intermediate shocks altogether. An anomaly appears in Friedrichs & Kranzer's later paper (1958), however, where it is wrongly stated that B_y may reverse in slow shocks for which $v_x \leqslant b_x$ on both sides.

Several other authors, including Helfer (1953), Lüst (1955) and Bazer & Ericson (1959), have performed extensive numerical explorations of the shocks that occur for the case of a perfect gas with $\gamma = \frac{5}{3}$. The parameters they choose to specify their shocks, though they may be convenient for computation, are awkward for classifying the shocks, chiefly owing to the fact that upstream and downstream conditions are not treated symmetrically. One might argue in their favour that one usually knows conditions upstream of a shock, but even this is debatable. Usually the obliquity of a shock is *not* known. Here the work of Cabannes (1959) is perhaps more realistic, since he takes flow deflexion angle as a known quantity instead.

Helfer recognized the *three* classes of shocks, fast, slow and intermediate, but did not distinguish the four types of intermediate shock. He presents his numerical results without much comment. Lüst's results are discussed later.

Bazer & Ericson chose as basic parameters to specify a shock the upstream values of $(a/b)^2$, $\tan^{-1}(B_y/B_x)$ and $|\Delta B_y|/B$, which they denote by s_0, θ_0 and h, respectively. Δ denotes change across the shock. They classify shocks into fast shocks in which B_y increases and 'slow' shocks in which B_y decreases *and may reverse*. We add the inverted commas to distinguish true slow shocks from 'slow' shocks, which also include the four kinds of intermediate shocks. They further subdivide their fast and 'slow' shocks into two types, depending on whether the shock is uniquely or doubly defined by h, for given values of s_0 and θ_0. This is a purely mathematical distinction, depending on the fickle behaviour of the variable h, although for 'slow' shocks it does distinguish between shocks in which $|B_y|$ increases or decreases, as is discussed later.

It is not difficult to locate the five different sorts of 'slow' shocks on Bazer & Ericson's graphs. Figure 8 shows a typical specimen, a rough and augmented version of the curves of v_x/b_x (upstream of shock) against h for $s_0 = \frac{1}{16}$ and $\theta_0 = 20°$ and $60°$, taken from their figure 7. Bazer & Ericson themselves point out that at A and F the shock degenerates to a slow sound wave and to an Alfvén shock, respectively. They also point out that D, the maximum of v_x/b_x (upstream) for given values of s_0 and θ_0, occurs when the flow is slow-sonic, downstream.

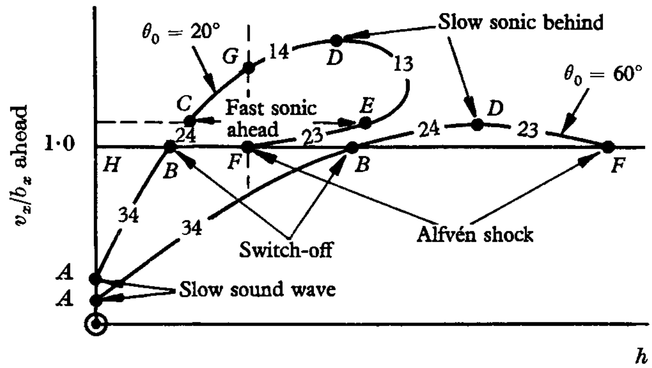

FIGURE 8. Approximate reproduction of Bazer and Ericson's graph. (N.B. h is $|\Delta B_y|/B$, not enthalpy.)

Another critical point on the curves is where $v_x = b_x$ (upstream) at B. Here we have the switch-off shock, where $h = \sin\theta_0$, so that $B_y = 0$ downstream. $HB = BF$. Finally, we have points where the flow is fast-sonic upstream. At such points, (23) gives

$$(v_x/b_x)^2_{\text{upstream}} = [(1+s_0)+\{(1+s_0)^2-4s_0\cos^2\theta_0\}^{\frac{1}{2}}]/2\cos^2\theta_0.$$

These points, C and E, are readily located on the $\theta_0 = 20°$ curve at the value $v_x/b_x = 1{\cdot}07$, but $\theta_0 = 60°$ is too large for them to occur. Here then is a more significant division of Bazer & Ericson's curves for 'slow' shocks into two types, depending on whether or not there occur these fast-sonic points. This division does not coincide with Bazer & Ericson's division even for a perfect gas.

The critical points A to F separate the five types of 'slow' shock in figure 8. Starting from the slow sound wave at A we traverse a region of pure slow shocks 34 before these change to intermediates of type 24 as the upstream state crosses the $v = b$, $\tau = \tau_0$ line at the switch-off shock B, and so-on, until the Alfvén shock F is reached.

As θ_0 tends to zero, the arc AD tends to the axis $h = 0$ representing ordinary shocks, EF vanishes and DE becomes the switch-on shock curve shown by Bazer & Ericson. Fast and slow shock régimes are connected to intermediate régimes via switch-on and switch-off shocks, respectively.

For each suitable value of s_0 there will be a value of θ_0 for which C, D and E coincide. This triple point then represents those freak intermediate shocks which are fast-sonic upstream and slow-sonic downstream.

Another interesting feature of Bazer & Ericson's curves is that they show when $|B_y|$ increases in intermediate shocks. This occurs on the part of the curve to the right of FG in figure 8. This question was also considered by Polovin & Liubarskii (1958), whose condition $B_x^2 < \mu v_{x1}^2 \rho_1(\rho_1+\rho_2)/2\rho_2$ in our notation is more simply written $\tau_0 < \bar{\tau}$ (bar denotes mean value between two sides of shock),

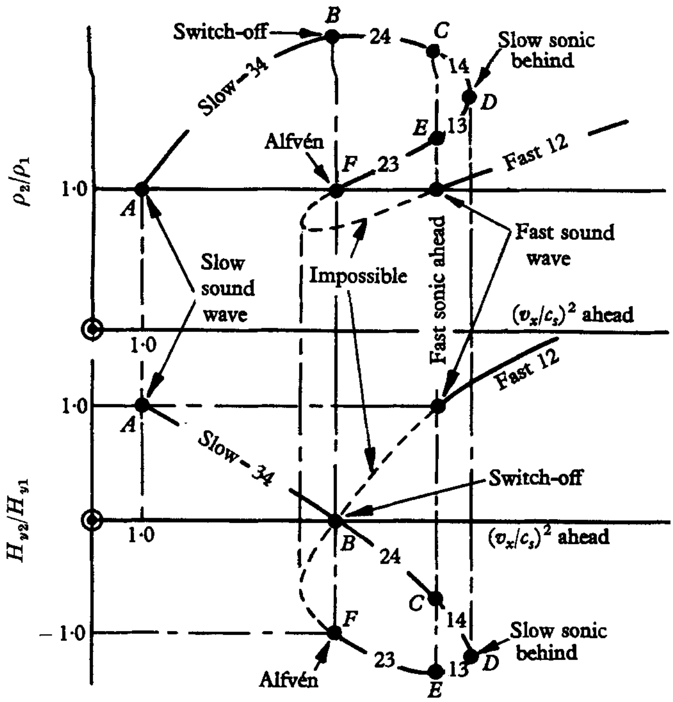

FIGURE 9. Approximate reproduction of Lüst's graphs.

which follows easily from (22). We observe that there exist shocks (G) besides Alfvén shocks (F) in which the magnetic field intensity is unchanged. Such shocks have various simple properties, including

$$\bar{\tau} = \tau_0, \quad \Delta p + G^2 \Delta\tau = 0 \quad \text{and} \quad \Delta h + (B^2/2\mu)\,\Delta\tau = 0.$$

Bazer & Ericson's graphs do not show the strong shocks considered in § 4.2.4, for which s_0 tends to zero. The only strong shocks they do show are *ordinary* strong shocks which occur as a case of fast shocks as a and b tend to zero, s_0 being finite and non-zero.

Lüst's (1955) graphs are similar to those of Bazer & Ericson. Figure 9 shows approximate specimens taken from Lüst's figures 5 and 6. The curves again correspond to constant values of s_0 and θ_0. Indeed the lower figure is almost the same as our figure 8, because $(v_x/b_x)^2 \propto (v_x/c_s)^2$ and $h \propto 1 - H_{y2}/H_{y1}$ for given values of s_0 and θ_0. The various regions on Lüst's curves are readily identifiable. In the example illustrated the intermediate branch enters the 13 and 14 regions by going fast-sonic ahead at the same value of $(v_x/c_s)^2$ as that at which the *fast* branch begins. Lüst's curves show how the fast and intermediate branches are connected via the impossible region, which infringes the Second Law. This part of the curve would be significant, however, if we were to interchange upstream and downstream conditions.

Bazer & Ericson remark that if the values of s_0 and θ_0 are fixed, Δs and v_x/b_x (upstream) are stationary when the normal velocity is sonic downstream. This identifies the point D in figures 8 and 9. This result is a general one, independent of the perfect gas assumption, as may be seen from the following argument.

To fix s_0 and θ_0, let us specify the upstream thermodynamic state and field strength and orientation completely, leaving the velocity free to vary. If suffix 1 denotes upstream, then we find that, as the upstream condition varies,

$$dF_x = 2v_{x1}dG, \quad dF_y = 2v_{y1}dG, \quad dH = (v_{x1}^2+v_{y1}^2)\,dG/G.$$

Generalizing (11) to permit variation of the spatially uniform parameter G gives

$$v_x dF_x + v_y dF_y + GT\,ds - G\,dH = (v_x^2+v_y^2)\,dG,$$

which may be applied to the downstream state (2), which has the same values of F_x, F_y, H and G as does (1). Hence

$$GT_2 ds_2 = \{(v_{x1}-v_{x2})^2+(v_{y1}-v_{y2})^2\}\,dG,$$

which is equivalent to Bazer & Ericson's equation (78). Hence if ds_2 vanishes so do dG, dF_x, dF_y and dH if the states 1 and 2 are different. But

$$\frac{dH}{d\rho_2} = \left(\frac{\partial H}{\partial \rho_2}\right)_{GF_y s_2} + \left(\frac{\partial H}{\partial s_2}\right)_{\rho_2 GF_y}\frac{ds_2}{d\rho_2} + \left(\frac{\partial H}{\partial G}\right)_{\rho_2 s_2 F_y}\frac{dG}{d\rho_2} + \left(\frac{\partial H}{\partial F_y}\right)_{\rho_2 s_2 G}\frac{dF_y}{d\rho_2}$$

and
$$\left(\frac{\partial H}{\partial s_2}\right)_{\rho_2 GF_y}, \quad \left(\frac{\partial H}{\partial G}\right)_{\rho_2 s_2 F_y} \quad \text{and} \quad \left(\frac{\partial H}{\partial F_y}\right)_{\rho_2 s_2 G}$$

are easily shown to be finite if $\tau_2 \neq \tau_0$. Thus, when s_2 and G are stationary we have

$$\left(\frac{\partial H}{\partial \rho_2}\right)_{GF_y s_2} = 0,$$

which, by table 1, implies that state 2 is sonic, provided $\tau_2 \neq \tau_0$. Moreover, since s_1, ρ_1 and b_1 are fixed, Δs and $(v_x/b_x)_1$ are indeed stationary in this case.

4.4. *Conclusion*

It is apparent that the generalized Rayleigh line offers many advantages for the study of magnetogasdynamic shocks, without the use of the perfect gas assumption. From it, we easily demonstrate the facts that the shocks must be compressive and that there are up to four states mutually accessible via six kinds of shocks, these four states being distinguished by the magnitude of the normal velocity in relation to the three local velocities, the two sound speeds and the Alfvén speed. It is easy to see how the six kinds of shock are connected with one another via the degenerate special cases of sonic states, Alfvén shocks, switch-on, switch-off and ordinary shocks. Not least of the advantages of the

Rayleigh line is its utility as a visual aid to remembering the multiplicity of cases which occur!

In this paper the questions of shock structure, shock stability and the ability of dissipative processes to permit the shocks to occur have been ignored. The most exhaustive discussion of shock structure is that of Germain (1959). Probably not all the shocks are possible; Alfvén *shocks*, as distinct from Alfvén *simple waves* (so broad that dissipation may be neglected) involve the absurdity of zero entropy change in the face of high spatial gradients, and one suspects that the closely related 23 shocks also fail because the entropy change is too small.

It is a pleasure to record my debt to M. D. Cowley for many stimulating discussions.

Appendix: Proof that the kinetic energy is successively less at the points 1*a*, 2*b*, 3*c* and 4*d* in figure 7

At 4*d* v_x and v_y both vanish and so we need consider 4*d* no further.

At 1*a*, 2*b* and 3*c*, τ is given by the cubic

$$E \equiv (\tau-\tau_x)(\tau-\tau_0)^2+\tau_0\tau_y^2/2 = 0, \tag{25}$$

derived from (21), where $\tau_x = F_x/G^2$, $\tau_y = F_y/G^2$. The kinetic energy

$$\tfrac{1}{2}(v_x^2+v_y^2) = \tfrac{1}{2}G^2\tau^2\{1+2(\tau_x-\tau)/\tau_0\},$$

in which we have used (25) to eliminate τ_y or F_y. We are thus essentially interested in the quantity $K = \tau^2\{1+2(\tau_x-\tau)/\tau_0\}$. Now

$$\begin{aligned}(\tau_x-\tau_0)K &= (\tau_x-\tau_0)K+(1+2\tau_x/\tau_0)E \\ &\equiv 3\tau^3-3\tau^2(\tau_0+2\tau_x)+\tau(\tau_0+2\tau_x)^2+(\tau_0+2\tau_x)(\tau_y^2/2-\tau_0\tau_x).\end{aligned}$$

This cubic form has been arranged to increase monotonically with τ. Its derivative is positive and only falls to zero at the value $\tau_f = \tfrac{1}{3}(\tau_0+2\tau_x)$, which always lies between 1*a* and 2*b*, and in fact is the value of τ at which 1*a* and 2*b* finally coincide when the fast branch falls below the τ axis as F_y increases. Since $\tau_x > \tau_0$ here, K therefore increases monotonically with τ at the three points 3*c*, 2*b* and 1*a* and our proof is complete.

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